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1994 J. Phys.: Condens. Matter 6 9329

(http://iopscience.iop.org/0953-8984/6/44/013)

J. Phys.: Condens. Matter 6 (1994) 9329–9333. Printed in the UK

A renormalization group approach to the Potts model on one-dimensional quasilattices

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Received 3 February 1994, in final form 7 July 1994

Abstract. A decimation renormalization is constructed and used to produce an exact RG recursion relation for the Potts model on one-dimensional quasilattices. The geometrical properties and critical properties of some one-dimensional quasilattices are discussed.

1. Introduction

The experimental discovery by Schechtman *et al* [1] of a metallic solid phase of Al–Mn alloy with icosahedral symmetry has considerably revived interest in quasicrystals. The study of one-dimensional models already reveals a large variety of consequences of the lack of periodicity in these systems [2–4]. It is interesting to investigate the effects of quasiperiodicity and self-similarity on the phase transition and the critical phenomena. Recently, one-dimensional quasicrystals have attracted much attention in the study of statistical mechanics of phase transitions [5–8], because classical spin models, such as the Ising and Potts models, are soluble for these lattices. Some very powerful approaches, such as transfer matrix techniques and the decimation renormalization method were used.

In this paper we report a decimation renormalization group treatment for the q-state Potts model (the Ising model is a special case of $q = 2$) on one-dimensional quasilattices. The self-similarity properties of quasilattices make the RG method quite an effective method of treatment. Furthermore the method can also provide an efficient method of numerical calculation with finite magnetic field. In addition, the renormalization relation may be used to obtain the Yang–Lee singularities [10].

2. Definition of model

A general one-dimensional quasilattice can be generated from a finite set of basic cells by a generalized induction procedure. We define the original pattern as stage 0 of the sequence, then stage $n + 1$ is obtained inductively from stage n by the following substitution rule: $\boldsymbol{a} \to \mathbf{T}\boldsymbol{a}$ where $\boldsymbol{a}$ represents a column vector: $\boldsymbol{a} = (\mathrm{A}, \mathrm{B})^{\mathrm{T}}$; and $\mathbf{T} = (t_{ij})$ is a 2×2 matrix with non-negative integer entries. Matrix $\mathbf{T}$ and its successive applications fully determine the sequence. Several examples follow.

Rule 1: the inflation rule is A → AB and B → A

$$\mathbf{T}_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \mathrm{A} \to \mathrm{AB} \to \mathrm{ABA} \to \ldots$$

Rule 2:

$$\mathbf{T}_2 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \Rightarrow \mathrm{A} \to \mathrm{ABB} \to \mathrm{ABBAA} \to \dots$$

Rule 3:

$$\mathbf{T}_3 = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \Rightarrow \mathrm{A} \to \mathrm{AB} \to \mathrm{ABAA} \to \dots$$

Rule 4:

$$\mathbf{T}_4 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \Rightarrow \mathrm{A} \to \mathrm{ABB} \to \mathrm{ABBABAB} \to \dots.$$

The above four cases are non-periodic structures, but the structures generated are nonetheless fully deterministic and highly ordered. As the length of the pattern goes to infinity, the ratio between the total number of different elements approaches a constant value. By solving the characteristic polynomial $P_{\mathrm{T}}(x)$ associated with the inflation rules $\mathbf{T}_1 \to \mathbf{T}_4$, we get four sets of eigenvalues:

$$\mathbf{T}_1 : \left\{(1+\sqrt{5})/2,\ (1-\sqrt{5})/2\right\};\ \mathbf{T}_2 : \{2, -1\};\ \mathbf{T}_3 : \{2, -1\};\ \mathbf{T}_4 : \left\{1+\sqrt{2},\ 1-\sqrt{2}\right\}.$$

It is noticed that the characteristic polynomial $P_{\mathrm{T}}(x)$ has only one root greater than one in absolute value for rules 1 and 4. The one-dimensional lattices defined by these two rules can be obtained by the projection method, and are strictly quasiperiodic.

In our model, we only consider the interaction between nearest-neighbour spins, and zero field is applied. Two kinds of coupling parameters K_{A} and K_{B} are introduced. The q-state Potts model Hamiltonian on the quasilattices can be written as follows:

$$-\frac{H}{KT} = \sum_{ij=\mathrm{NN}} K_{ij}\delta_{\sigma_i\sigma_j} \qquad (\mathrm{NN} = \text{nearest neighbour}).$$

$K_{ij} = K_{\mathrm{A}}$ if i and j are connected by bond A, $K_{ij} = K_{\mathrm{B}}$ if i and j are connected by bond B, where K_{ij} is the 'coupling' between the site spins i and j; σ_i is the Potts spin associated with site i and takes q possible values ($\sigma_i = 1, 2, \dots, q$), $\sigma_{\sigma_i\sigma_j}$ is the Kronecker function.

3. Critical points and critical exponents

Let us define two matrices as follows:

$$\begin{aligned} \langle\sigma|\hat{A}|\sigma'\rangle &= \exp\left(K_{\mathrm{A}}\delta_{\sigma\sigma'}\right) \\ \langle\sigma|\hat{B}|\sigma'\rangle &= \exp\left(K_{\mathrm{B}}\delta_{\sigma\sigma'}\right). \end{aligned} \tag{1}$$

Obviously the two matrices are symmetric about their diagonal and commute with one another, and the product of these matrices still keeps the above properties. By means of equation (1), we get the following expressions which we will use later:

$$\begin{aligned} &[\hat{A}]_{\alpha\alpha} = \mathrm{e}^{K_{\mathrm{A}}} \qquad [\hat{A}]_{\alpha\beta} = 1 \\ &\left[\hat{A}^2\right]_{\alpha\alpha} = \mathrm{e}^{2K_{\mathrm{A}}} + q - 1 \qquad \left[\hat{A}^2\right]_{\alpha\beta} = 2\mathrm{e}^{K_{\mathrm{A}}} + q - 2 \\ &[\hat{A}\hat{B}]_{\alpha\alpha} = \mathrm{e}^{K_{\mathrm{A}}+K_{\mathrm{B}}} + q - 1 \qquad [\hat{A}\hat{B}]_{\alpha\beta} = \mathrm{e}^{K_{\mathrm{A}}} + \mathrm{e}^{K_{\mathrm{B}}} + q - 2 \\ &\left[\hat{A}\hat{B}^2\right]_{\alpha\alpha} = \mathrm{e}^{K_{\mathrm{A}}}\left(\mathrm{e}^{2K_{\mathrm{B}}} + q - 1\right) + (q-1)\left(2\mathrm{e}^{K_{\mathrm{B}}} + q - 2\right) \\ &\left[\hat{A}\hat{B}^2\right]_{\alpha\beta} = \mathrm{e}^{K_{\mathrm{A}}}\left(2\mathrm{e}^{K_{\mathrm{B}}} + q - 2\right) + \left(\mathrm{e}^{2K_{\mathrm{B}}} + q - 1\right) + (q-2)\left(2\mathrm{e}^{K_{\mathrm{B}}} + q - 2\right). \end{aligned}$$

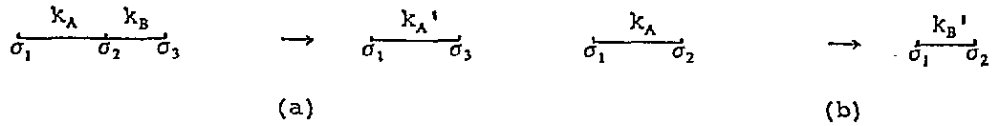

Figure 1. A schematic RG transformation for rule 1. (a) Decimating σ_2 and leaving σ_1 and σ_3 fixed in unit 1; (b) leaving σ_1 and σ_2 fixed in unit 2.

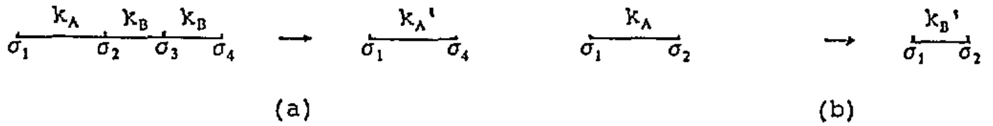

Figure 2. A schematic RG transformation for rule 2. (a) Decimating σ_2 and σ_3, leaving σ_1 and σ_4 fixed in unit 1; (b) leaving σ_1 and σ_2 fixed in unit 2.

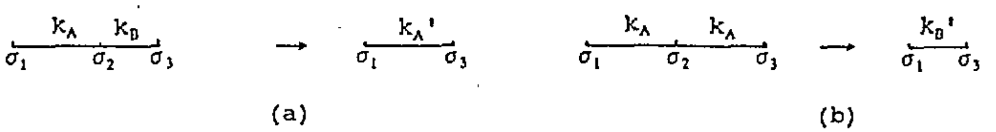

Figure 3. A schematic RG transformation for rule 3. (a) Decimating σ_2 and leaving σ_1 and σ_3 fixed in unit 1; (b) decimating σ_2 and leaving σ_1 and σ_3 fixed in unit 2.

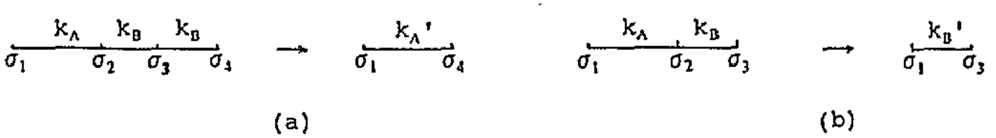

Figure 4. A schematic RG transformation for rule 4. (a) Decimating σ_2 and σ_3, leaving σ_1 and σ_4 fixed in unit 1; (b) decimating σ_2 and leaving σ_1 and σ_3 fixed in unit 2.

To construct recursion relations we consider the two basic units as the components of the original nth construction stage. By integrating over the spins on some sites and leaving the remainder fixed, we obtain a renormalized construction which has the components of the $(n-1)$th construction stage. The RG procedures are shown schematically in figures 1–4. They respectively correspond to the following expressions:

$$\mathbf{T}_1 \begin{cases} F\exp\left(K_A'\delta_{\sigma_1\sigma_3}\right) = \sum_{\sigma_2}\exp\left(K_A\delta_{\sigma_1\sigma_2} + K_B\delta_{\sigma_2\sigma_3}\right) = \langle\sigma_1|\hat{A}\hat{B}|\sigma_3\rangle \\ F'\exp\left(K_B'\delta_{\sigma_1\sigma_2}\right) = \exp\left(K_A\delta_{\sigma_1\sigma_2}\right) = \langle\sigma_1|\hat{A}|\sigma_2\rangle \end{cases}$$

$$\mathbf{T}_2 \begin{cases} F\exp\left(K_A'\delta_{\sigma_1\sigma_4}\right) = \sum_{\sigma_2\sigma_3}\exp\left(K_A\delta_{\sigma_1\sigma_2} + K_B\delta_{\sigma_2\sigma_3} + K_B\delta_{\sigma_3\sigma_4}\right) = \langle\sigma_1|\hat{A}\hat{B}^2|\sigma_4\rangle \\ F'\exp\left(K_B'\delta_{\sigma_1\sigma_2}\right) = \exp\left(K_A\delta_{\sigma_1\sigma_2}\right) = \langle\sigma_1|\hat{A}|\sigma_2\rangle \end{cases}$$

$$\mathbf{T}_3 \begin{cases} F\exp(K'_{\mathrm{A}}\delta|_{\sigma_1\sigma_3}) = \sum_{\sigma_2}\exp(K_{\mathrm{A}}\delta_{\sigma_1\sigma_2} + K_{\mathrm{B}}\delta_{\sigma_2\sigma_3}) = \langle\sigma_1|\hat{A}\hat{B}|\sigma_3\rangle \\ F'\exp(K'_{\mathrm{B}}\delta_{\sigma_1\sigma_3}) = \sum_{\sigma_2}\exp(K_{\mathrm{A}}\delta_{\sigma_1\sigma_2} + K_{\mathrm{A}}\delta_{\sigma_2\sigma_3}) = \langle\sigma_1|\hat{A}^2|\sigma_3\rangle \end{cases}$$

$$\mathbf{T}_4 \begin{cases} F\exp(K'_{\mathrm{A}}\delta_{\sigma_1\sigma_4}) = \sum_{\sigma_2\sigma_3}\exp(K_{\mathrm{A}}\delta_{\sigma_1\sigma_2} + K_{\mathrm{B}}\delta_{\sigma_2\sigma_3} + K_{\mathrm{B}}\delta_{\sigma_3\sigma_4}) = \langle\sigma_1|\hat{A}\hat{B}^2|\sigma_4\rangle \\ F'\exp(K'_{\mathrm{B}}\delta_{\sigma_1\sigma_3}) = \sum_{\sigma_2}\exp(K_{\mathrm{A}}\delta_{\sigma_1\sigma_2} + K_{\mathrm{B}}\delta_{\sigma_2\sigma_3}) = \langle\sigma_1|\hat{A}\hat{B}|\sigma_3\rangle \end{cases}$$

and give rise to the recursion relations as follows:

$$\mathbf{T}_1 : \begin{cases} \mathrm{e}^{K'_{\mathrm{A}}} = [\hat{A}\hat{B}]_{\alpha\alpha}/[\hat{A}\hat{B}]_{\alpha\beta} = (\mathrm{e}^{K_{\mathrm{A}}+K_{\mathrm{B}}} + q - 1)/(\mathrm{e}^{K_{\mathrm{A}}} + \mathrm{e}^{K} + q - 2) \\ \mathrm{e}^{K'_{\mathrm{B}}} = [\hat{A}]_{\alpha\alpha}/[\hat{A}]_{\alpha\beta} = \mathrm{e}^{K_{\mathrm{A}}} \end{cases} \tag{2}$$

$$\mathbf{T}_2 : \begin{cases} \mathrm{e}^{K'_{\mathrm{A}}} = [\hat{A}\hat{B}^2]_{\alpha\alpha}/[\hat{A}\hat{B}^2]_{\alpha\beta} = [\mathrm{e}^{K_{\mathrm{A}}}(\mathrm{e}^{2K_{\mathrm{B}}} + q - 1) + (q-1)(2\mathrm{e}^{K_{\mathrm{B}}} + q - 2)] \\ \mathrm{e}^{K'_{\mathrm{B}}} = [\hat{A}]_{\alpha\alpha}/[\hat{A}]_{\alpha\beta} = \mathrm{e}^{K_{\mathrm{A}}} \end{cases} \tag{3}$$

$$\mathbf{T}_3 : \begin{cases} \mathrm{e}^{K'_{\mathrm{A}}} = [\hat{A}\hat{B}]_{\alpha\alpha}/[\hat{A}\hat{B}]_{\alpha\beta} = (\mathrm{e}^{K_{\mathrm{A}}+K_{\mathrm{B}}} + q - 1)/(\mathrm{e}^{K_{\mathrm{A}}} + \mathrm{e}^{K_{\mathrm{B}}} + q - 2) \\ \mathrm{e}^{K'_{\mathrm{B}}} = [\hat{A}^2]_{\alpha\alpha}/[\hat{A}^2]_{\alpha\beta} = (\mathrm{e}^{2K_{\mathrm{A}}} + q - 1)/(2\mathrm{e}^{K_{\mathrm{A}}} + q - 2) \end{cases} \tag{4}$$

$$\mathbf{T}_4 : \begin{cases} \mathrm{e}^{K'_{\mathrm{A}}} = [\hat{A}\hat{B}^2]_{\alpha\alpha}/[\hat{A}\hat{B}^2]_{\alpha\beta} = [\mathrm{e}^{K_{\mathrm{A}}}(\mathrm{e}^{2K_{\mathrm{B}}} + q - 1) + (q-1)(2\mathrm{e}^{K_{\mathrm{B}}} + q - 2)] \\ [\mathrm{e}^{K_{\mathrm{A}}}(2\mathrm{e}^{K_{\mathrm{B}}} + q - 2) + (\mathrm{e}^{2K_{\mathrm{B}}} + q - 1) + (q-2)(2\mathrm{e}^{K_{\mathrm{B}}} + q - 2)] \\ \mathrm{e}^{K'_{\mathrm{B}}} = [\hat{A}\hat{B}]_{\alpha\alpha}/[\hat{A}\hat{B}]_{\alpha\beta} = (\mathrm{e}^{K_{\mathrm{A}}+K_{\mathrm{B}}} + q - 1)/(\mathrm{e}^{K_{\mathrm{A}}} + \mathrm{e}^{K_{\mathrm{B}}} + q - 2) \end{cases} . \tag{5}$$

It is clear that $K^*_{\mathrm{A}} = K^*_{\mathrm{B}} = \infty$ is one of the fixed points, because the total order of powers of e^x in the numerical is higher than in the denominator where e^x denotes $\mathrm{e}^{K_{\mathrm{A}}}$ and $\mathrm{e}^{K_{\mathrm{B}}}$. The recursion relations have other fixed points $\mathrm{e}^{K^*_{\mathrm{A}}} = \mathrm{e}^{K^*_{\mathrm{B}}}$: $\mathbf{T}_1 : \{1,\ 1-q\}$; $\mathbf{T}_2 : \{1,\ (2-q)/2,\ 1-q\}$; $\mathbf{T}_3 : \{1,\ 1-q\}$; $\mathbf{T}_4 : \{1,\ 1-q\}$.

Originally q was defined as the number of spin states and has to be an integer; here q is interpreted as a material parameter which may be varied arbitrarily.

The point $\mathrm{e}^{K^*} = 1$ corresponds to $T = \infty$, which is an infinite-temperature stable fixed point, and the point $\mathrm{e}^{K^*} = \infty$ to $T = 0$, which is a zero-temperature unstable fixed point. The domain of attraction $D(1)$ collects all starting points in parameter e^k space which come arbitrarily close to unity after definite renormalization transformation. In the spirit of renormalization theory $D(1)$ is identified as the high-temperature phase, whereas $D(\infty)$ stands for the magnetic low-temperature phase. As regards points $1-q$ AND $(2-Q)/2$, they have a special role in discussing Yang–Lee singularities [10]. The corresponding renormalization equations (2)–(5) are technically rational mapping of the complex plane when the temperature T is extended from the real axis into the complex plane. The theory of iteration of such mapping was established around 1920 by Julia [11] and Fatou [12]. This establishes the remarkable connection between phase transitions and Julia sets, which will help us to understand the nature of the Yang–Lee singularities. But the physical interpretation is still an open question.

Here we discuss the critical properties at zero temperature. By calculating the linear recursion relations around the zero-temperature fixed points, we obtained the four renormalized matrices:

$$\mathbf{R}_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{R}_2 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{R}_3 = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \qquad \mathbf{R}_4 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

Compared with the substitution matrices $\mathbf{T}_1$–$\mathbf{T}_4$, it is shown that these renormalization matrices are exactly equal to their substitution matrices.

The critical correlation length exponent is related to the largest eigenvalue, $\nu^{-1} = (\ln\lambda)/\ln b)$ where b is the geometrical scaling factor and λ is the eigenvalue of the renormalized matrices. The renormalization group transformation is a decimation transformation, which mimics the deflation rule. The rescaling factor b is equal to the positive eigenvalue of the substitution matrix $\mathbf{T}_1$–$\mathbf{T}_4$, respectively. Since the renormalized matrices are exactly equal to their substitution matrices, ν is equal to unity.

4. Conclusion

We have performed a decimation renormalization for the Potts model on one-dimensional two-tile non-periodic lattices in the absence of a magnetic field. The recursion equations are produced and the immediate calculations show that the system has no phase transition at finite temperature. The properties of critical points and critical exponents are discussed.

Acknowledgments

Support from National Natural Science Foundation of China and the Foundation for PhD Graduates of the National Committee of Education are acknowledged.

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